# Some algebra for ringdown measurements

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We begin with the equations in Isogai et al.[1] with a modification for $m_1$:

$$\begin{aligned} m_1 &= R_1(P_0 + P_1) && (1)\\ m_2 &= P_0 K T_1^2 R_2 && (2)\\ m_3 &= P_0 K[R_1^{1/2} - R_2^{1/2}(T_1 + R_1)]^2 + P_1 && (3)\\ m_4 &= \tau/2, && (4) \end{aligned}$$

with $\tau/2 = -1/[f_{\mathrm{FSR}} \ln(R_1 R_2)]$ and $K = 1/[1 - (R_1 R_2)^{1/2}]^2$. We want to find an expression for $T_1$ in terms of known quantities. By assuming a value for $R_2$, we can use $m_4$ to fix the values of $K$ and $R_1$. Further, using $P_1 = m_1/R_1 - P_0$, we have

$$\begin{aligned} m_2 &= P_0 K T_1^2 R_2 && (5)\\ m_3 &= P_0 K[R_1^{1/2} - R_2^{1/2}(T_1 + R_1)]^2 + m_1/R_1 - P_0 && (6)\\ &= P_0 K[R_1 - 2(R_1 R_2)^{1/2}(T_1 + R_1) + R_2(T_1 + R_1)^2] + m_1/R_1 - P_0 && (7)\\ &= \frac{m_2}{T_1^2 R_2}[R_1 - 2(R_1 R_2)^{1/2}(T_1 + R_1) + R_2(T_1 + R_1)^2] + m_1/R_1 - \frac{m_2}{K T_1^2 R_2} && (8)\\ \Rightarrow \quad m_3 R_2 T_1^2 &= m_2[R_1 - 2(R_1 R_2)^{1/2}(T_1 + R_1) + R_2(T_1 + R_1)^2] + m_1(R_2/R_1)T_1^2 - m_2/K && (9)\\ &= m_2[R_1 - 2R_1^{3/2}R_2^{1/2} + R_2 R_1^2 - 2R_1^{1/2}R_2^{1/2}T_1 + 2R_1 R_2 T_1 + R_2 T_1^2]\\ &\quad + m_1(R_2/R_1)T_1^2 - m_2/K && (10)\\ \Rightarrow \quad 0 &= (-m_3 + m_2 + m_1/R_1)R_2 T_1^2 + 2m_2(R_1 R_2 - R_1^{1/2}R_2^{1/2})T_1\\ &\quad + m_2(R_1 - 2R_1^{3/2}R_2^{1/2} + R_1^2 R_2 - 1/K) && (11)\\ &= [1 + m_1/(m_2 R_2) - m_3/m_2]R_2 T_1^2 + 2(R_1 R_2 - R_1^{1/2}R_2^{1/2})T_1\\ &\quad + R_1(1 - R_1^{1/2}R_2^{1/2})^2 - 1/K. && (12) \end{aligned}$$

Since this equation is quadratic in $T_1$, the solution is straightforward.

[1] "Loss in long-storage-time optical cavities", *Optics Express*, 30114 (2013).