$$S = \frac{K_c}{1+if/f_c} \; \frac{f^2}{f^2 - i\frac{f f_s}{Q} + f_s^2}$$

low freq pcal line

For typical values of $f = 7.63\ Hz$, $f_s = 8\ Hz$ and $Q = 25$

$$\frac{f^2}{f^2 - i\frac{f f_s}{Q} + f_s^2} = 0.4762 + I 0.0095$$

However for $f = 331.9\ Hz$ where we run pcal line that is used to estimate cavity pole and optical gain

$$\frac{f^2}{f^2 - i\frac{f f_s}{Q} + f_s^2} = 0.9994 + I\ 0.0010$$

which is pretty close to one and so we can neglect that term when estimating cavity pole and optical gain.

[Note: Here by construction K_c, f_1, f_c are real quantities]

$$\Rightarrow \quad S_c \approx \frac{K_c}{1 + i f_1/f_c} = \frac{K_c}{1+\frac{f_1^2}{f_c^2}}\left[1 - i\frac{f_1}{f_c}\right] = S_{f_1}$$

$$\Rightarrow \quad Re[S_c] = \frac{K_c}{1+\frac{f_1^2}{f_c^2}} \; ; \; Im[S_c] = \frac{-K_c \cdot \frac{f_1}{f_c}}{1+\frac{f_1^2}{f_c^2}} \; ; \; |S_c| = \frac{K_c^2}{1+\frac{f_1^2}{f_c^2}}$$

$$\Rightarrow \quad \boxed{K_c = \frac{|S_c|^2}{Re[S_c]}} \quad \text{and} \quad \boxed{\frac{f_c}{f_1} = \frac{-Re[S_c]}{Im[S_c]}}$$

Substituting K_c and f_c in S for f_2 line, we get

$$S_{f_2} = \frac{K_c}{1 + i\frac{f_2}{f_c}} \cdot \frac{f_2^2}{f_2^2 - i\frac{f_2 f_s}{Q} + f_s^2}$$

$$\Rightarrow \quad \frac{S_{f_2}}{\frac{K_c}{1+i f_2/f_c}} = \frac{f_2^2}{f_2^2 - i\frac{f_2 f_s}{Q} + f_s^2} = S_s$$

$$S_s = \frac{f_2^2}{f_2^2 - i\frac{f_2 f_s}{Q} + f_s^2} \cdot \frac{f_2^2 + f_s^2 + i\frac{f_2 f_s}{Q}}{f_2^2 + f_s^2 + i\frac{f_2 f_s}{Q}}$$

[Note: By construction, f_2, f_s, Q are real quantities]

$$= \frac{f_2^2}{(f_2^2 + f_s^2)^2 + \frac{f_2^2 f_s^2}{Q^2}} \left(f_2^2 + f_s^2 + i\frac{f_2 f_s}{Q} \right)$$

$$\Rightarrow \mathrm{Re}[S_s] = \frac{f^2}{(f_2^2 + f_s^2)^2 + \frac{f_2^2 f_s^2}{Q^2}} (f_2^2 + f_s^2) \quad ; \quad \mathrm{Im}[S_s] = \frac{f_s f_2^3 / Q}{(f_2^2 + f_s^2)^2 + \frac{f_2^2 f_s^2}{Q^2}}$$

$$|S_s|^2 = \frac{f^4}{(f_2^2 + f_s^2)^2 + \frac{f_2^2 f_s^2}{Q^2}}$$

$$\Rightarrow \frac{\mathrm{Re}[S_c]}{|S_c|^2} = \frac{f_2^2 + f_s^2}{f_2^2} \quad ; \quad \frac{\mathrm{Re}[S_c]}{\mathrm{Im}[S_c]} = \frac{f_2^2 + f_s^2}{f_2 f_s} \cdot Q \quad ; \quad \frac{|S_c|^2}{\mathrm{Im}[S_c]} = \frac{f_2}{f_s} \cdot Q$$

$$\Downarrow$$

$$f_s^2 = \left[\frac{\mathrm{Re}[S_c]}{|S_c|^2} - 1 \right] f_2^2$$

$$\Rightarrow \boxed{f_s = f_2 \sqrt{\frac{\mathrm{Re}[S_c]}{|S_c|^2} - 1} = \frac{f_2}{|S_c|} \sqrt{\mathrm{Re}[S_c] - |S_c|^2}}$$

Substituting this back into $\frac{|S_c|^2}{\mathrm{Im}[S_c]}$, we get

$$\boxed{Q = \frac{f_s}{f_2} \cdot \frac{|S_c|^2}{\mathrm{Im}[S_c]} = \frac{|S_c|}{\mathrm{Im}[S_c]} \sqrt{\mathrm{Re}[S_c] - |S_c|^2}}$$

We measure (or can ~~estimate~~ calculate) S_{f_1} and S_{f_2} and from them we can get S_c and S_c. We can then use S_s and S_c to get K_c, f_c, f_s and Q.

$$\left[S_c = S_{f_1} \quad ; \quad S_s = \frac{S_{f_2}}{\frac{K_c}{1 + i f_2/f_c}} \right]$$