Why is the AS72 signal insensitive to linear order longitudinal motions?

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1 Longitudinally unperturbed fields

We consider a coordinate system rotating at $\exp(-2\pi \mathrm{i} f_0 t)$ where f_0 is the main carrier frequency. We further set the real and imaginary axes in such a way that the main carrier field is always real-positive, and we implicitly imply each field's temporal variation with a subscript, i.e., $E_{f_{\rm sb}} \equiv |E_{f_{\rm sb}}| \exp\left(-2\pi \mathrm{i} f_{\rm sb} t\right)$.

In this note we keep track of 5 fields, the carrier, and the upper and lower 45 and 118 MHz sidebands. Note that the RF propagation phase will only rotate the upper and lower sidebands symmetrically about the imaginary axis. This is also true for the cavity filtering when the carrier is exactly on-resonance or anti-resonance; the phase delay to the upper and lower sidebands will be symmetric about the imaginary axis, while the amplitude attenuation will be the same for the upper and lower sidebands. Therefore, we can generically write

$$E_c = a_c, \tag{1}$$

$$E_{+45} = a_{45} + \mathrm{i} b_{45}, \qquad E_{-45} = -a_{45} + \mathrm{i} b_{45}, \tag{2}$$

$$E_{+118} = a_{118} + \mathrm{i} b_{118}, \quad E_{-118} = -a_{118} + \mathrm{i} b_{118}, \tag{3}$$

where the fields (E) are complex and $a, b \in \mathbb{R}$. See also Fig. 1.

Consider the AS45 signal first. This is derived from a traditional $1f$ PDH pair where we expect the power vanishes exactly at 45 MHz, $P_{45} \equiv 0$. Mathematically this can be seen as the following,

$$P_{45} = \mathrm{Re}\left[S_{45} \exp(-\mathrm{i}\phi_{\rm dmd})\right], \tag{4}$$

where P_{45} is the demodulated power at 45 MHz and $\phi_{\rm dmd}$ the demodulation phase. The quantity $S_{45} \in \mathbb{C}$ is given as

$$\begin{aligned} S_{45} &= E_c^* E_{45} + E_c E_{-45}^*, \\ &= a_c(a_{45} + \mathrm{i} b_{45}) + a_c(-a_{45} - \mathrm{i} b_{45}) = 0. \end{aligned} \tag{5}$$

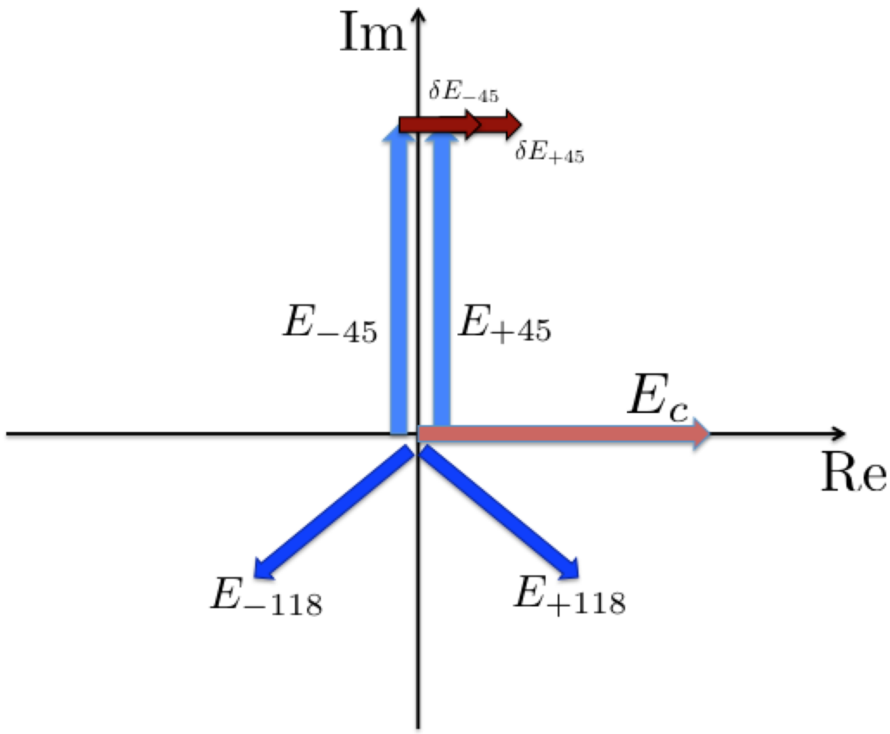

Figure 1: Phase diagram of the fields we consider in this note. The carrier field is placed along the positive real axis. The upper and lower RF sidebands are symmetric about the imaginary axis. A small longitudinal/phase perturbation on the 45 MHz RF sideband will create a non-vanishing beat note with the carrier field. However, the symmetrical axis of this perturbation field (along the real axis) is orthogonal to the symmetrical axis of the 118 MHz RF sideband (along the imaginary axis), and thus the beat note between these fields at 72 MHz will vanish identically zero, as perfect cancellation will happen after taking complex conjugation and summing signals together. See the text for a more rigorous proof.

This is exactly why a WFS signal derived from a $1f$ PDH pair will be insensitive to spot position.

On the other hand, the AS72 signal is effectively a $2f$ optical lever signal. This is a sideband-beating-sideband signal and is intrinsically different from the $1f$ carrier-beating-sideband signals such as the AS45. It further means that it has a non-vanishing power at 72 MHz, as

$$P_{72} = \mathrm{Re}\left[S_{72}\exp(-\mathrm{i}\phi_{\mathrm{dmd}})\right], \tag{6}$$

with

$$\begin{aligned} S_{72} =& E_{45}^* E_{118} + E_{-45}E_{-118}^*, \\ =& (a_{45} - \mathrm{i}b_{45})(a_{118} + \mathrm{i}b_{118}) + (-a_{45} + \mathrm{i}b_{45})(-a_{118} - \mathrm{i}b_{118}), \\ =& 2\left[(a_{45}a_{118} + b_{45}b_{118}) + \mathrm{i}(a_{45}b_{118} - b_{45}a_{118})\right] \neq 0. \end{aligned} \tag{7}$$

2 Small longitudinal/phase detuning

For small longitudinal detuning, it generates phase sidebands to the unperturbed fields, as

$$\delta E_c = \mathrm{i}\phi_c a_c, \tag{8}$$

$$\delta E_{+45} = \mathrm{i}\phi_{45}a_{45} - \phi_{45}b_{45}, \qquad \delta E_{-45} = -\mathrm{i}\phi_{45}a_{45} - \phi_{45}b_{45}, \tag{9}$$

$$\delta E_{+118} = \mathrm{i}\phi_{118}a_{118} - \phi_{118}b_{118}, \quad \delta E_{-118} = -\mathrm{i}\phi_{118}a_{118} - \phi_{118}b_{118}, \tag{10}$$

where we have used $\phi_f \in \mathbb{R}$ to denote the phase detuning for a specific field at frequency f.

Again we first consider the traditional $1f$ PDH signal AS45. We can decompose the detuned signal δS_{45} into two parts, $\delta S_{45}^{(\delta 45)} = E_c^*\delta E_{45} + E_c \delta E_{-45}^*$ denoting the perturbed 45 MHz sidebands beating the unperturbed carrier, and $\delta S_{45}^{(\delta c)} = (\delta E_c)^* E_{45} + (\delta E_c)E_{-45}^*$. We thus have

$$\begin{aligned} \delta S_{45}^{(\delta 45)} &= E_c^*\delta E_{45} + E_c\delta E_{-45}^*, \\ &= 2a_c(-\phi_{45}b_{45} + \mathrm{i}\phi_{45}a_{45}), \end{aligned} \tag{11}$$

and similarly for the $\delta S_{45}^{(\delta c)}$ term. This is why a $1f$ PDH signal can be used to sense longitudinal fluctuations.

However, the $2f$ oplev signal such as AS72 (and AS36) will not see this perturbation to the linear order. We can write $\delta S_{72} = \delta S_{72}^{(\delta 45)} + \delta S_{72}^{(\delta 118)}$, where

$$
\begin{aligned}
\delta S_{72}^{(\delta 45)} &= (\delta E_{45})^* E_{118} + (\delta E_{-45}) E_{-118}^*, \\
&= (-\phi_{45} b_{45} - \mathrm{i}\phi_{45} a_{45})(a_{118} + \mathrm{i} b_{118}) + (-\phi_{45} b_{45} - \mathrm{i}\phi_{45} a_{45})(-a_{118} - \mathrm{i} b_{118}), \\
&= \phi_{45} \Big[(-b_{45} a_{118} + a_{45} b_{118}) + \mathrm{i}(-a_{45} a_{118} - b_{45} b_{118}) \\
&\qquad + (b_{45} a_{118} - a_{45} b_{118}) + \mathrm{i}(a_{45} a_{118} + b_{45} b_{118}) \Big] \\
&= 0.
\end{aligned}
\tag{12}
$$

Similarly, it is easy to show that $\delta S_{72}^{(\delta 118)} = 0$.

Consequently, *the AS72 signal is insensitive to the longitudinal perturbation to the linear order*.