Cross correlation with imbalanced PDs

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1 Introduction

Documents to cite: [1] Following the notation in Figure 5.1 of [2], we can allow the DCPDs to have different responsivitities described by the transfer functions A and B. The splitting ratio of the OMC beamsplitter could potentially be different from 50/50, which can also be included in the factors A and B. Finally, there are real matrix elements used to combine the individual DCPD signals d_A and d_B into the sum channel $d_+ = M_A^+ d_A + M_B^+ d_B$, and a null channel $d_- = M_A^- d_A + M_B^- d_B$. In this notation the transfer function which is normally called the sensing function is:

$$C = Z\left[AM_A^+ + BM_B^+\right] \tag{1}$$

Allowing for these additions, the expression for the DARM OLG becomes:

$$G = YC \tag{2}$$

Where Z is the interferometer response in W/m and Y is the actuator response, including the digital filters of the DARM loop, in meter/counts of DCPD sum.

This loop uses the DCPD sum signal as an error signal:

$$\begin{aligned}
d_+ &= M_A^+ d_A + M_B^+ d_B \\
&= \left[(Yd_+ + n_c)ZB + n_B\right] M_B^+ + \left[(Yd_+ n_c)ZA + n_A\right] M_A^+ \\
&= Gd_+ + Cn_c + M_A^+ n_A + M_B^+ n_b \\
&= \frac{1}{1-G}\left(Cn_c + M_A^+ n_A + M_B^+ n_B\right)
\end{aligned} \tag{3}$$

where n_c is the correlated noise, n_A and n_B are the sensor noises of DCPDs, and d_A and d_B are the signals as recorded by the DCPD A and B channels, which are digitally recorded after the analog transfer function but before the matrix elements M_A^+ and M_B^+. The signal on each PD is given by:

$$\begin{aligned}
d_A &= d_+ YZA + n_c ZA + n_A \\
&= \frac{ZA}{1-G} n_c + \left(\frac{YZAM_A^+}{1-G} + 1 \right) n_A + \frac{YZAM_B^+}{1-G} n_B \\
&= \frac{ZA}{1-G} n_c + \frac{1 - YZBM_B^+}{1-G} n_A + \frac{YZAM_B^+}{1-G} n_B \\
d_B &= d_+ YZB + n_c ZB + n_B \\
&= \frac{ZB}{1-G} n_c + \frac{YZBM_A^+}{1-G} n_A + \left(\frac{YZBM_B^+}{1-G} + 1 \right) n_B \\
&= \frac{ZB}{1-G} n_c + \frac{YZBM_A^+}{1-G} n_A + \frac{1 - YZAM_A^+}{1-G} n_B
\end{aligned} \tag{4}$$

We can find the power spectral densities assuming n_A, n_B, and n_C are independent:

$$\begin{aligned}
\langle d_A, d_A \rangle &= \frac{|ZA|^2}{|1-G|^2} \langle n_c, n_c \rangle + \frac{|1 - YZBM_B^+|^2}{|1-G|^2} \langle n_A, n_A \rangle + \frac{|YZAM_B^+|^2}{|1-G|^2} \langle n_B, n_B \rangle \\
\langle d_B, d_B \rangle &= \frac{|ZB|^2}{|1-G|^2} \langle n_c, n_c \rangle + \frac{|YZBM_A^+|^2}{|1-G|^2} \langle n_A, n_A \rangle + \frac{|1 - YZAM_A^+|^2}{|1-G|^2} \langle n_B, n_B \rangle
\end{aligned} \tag{5}$$

The cross spectral density is:

$$\begin{aligned}
\langle d_A, d_B \rangle = & \frac{|Z|^2}{|(1-G)|^2} A^* B \langle n_c, n_c \rangle + \frac{YZBM_A^+ - |YZB|^2 M_A^+ M_B^+}{|1-G|^2} \langle n_A, n_A \rangle \\
& + \frac{Y^* Z^* A^* M_B^+ - |YZA|^2 M_A^+ M_B^+}{|1-G|^2} \langle n_B, n_B \rangle
\end{aligned} \tag{6}$$

2 Simplifications

We can make some simplifying assumptions. A and B differ only by a scalar gain (we are ignoring frequency dependent differences in the whitening chassis responses, and attributing the diference in the two gains to differences in PD responsivity or to the beam splitter ratio not being exactly 50%.) The ratio of these gains is the real scalar h, $(A = hB)$. The procedure for setting the DCPD matrix is described in [3], in particular $M_B^+ = 2 - M_A^+$. The sensing function becomes

$$C = ZB \left[2 + (h-1) M_A^+ \right] \tag{7}$$

Our expressions for the PSDs and cross spectral density becomes:

$$
\begin{aligned}
\langle d_A, d_A\rangle &= \frac{h^2|ZB|^2}{|1-G|^2}\langle n_c, n_c\rangle + \frac{|1-YZB(2-M_A^+)|^2}{|1-G|^2}\langle n_A, n_A\rangle + \frac{h^2(2-M_A^+)^2|YZB|^2}{|1-G|^2}\langle n_B, n_B\rangle \\
\langle d_B, d_B\rangle &= \frac{|ZB|^2}{|1-G|^2}\langle n_c, n_c\rangle + \frac{|YZBM_A^+|^2}{|1-G|^2}\langle n_A, n_A\rangle + \frac{|1-hYZBM_A^+|^2}{|1-G|^2}\langle n_B, n_B\rangle \\
\langle d_A, d_B\rangle &= \frac{h|ZB|^2}{|(1-G)|^2}\langle n_c, n_c\rangle + \frac{YZBM_A^+ - |YZB|^2(2M_A^+ - M_A^+2)}{|1-G|^2}\langle n_A, n_A\rangle \\
&\quad + \frac{hY^*Z^*B^*(2-M_A^+) - h^2|YZb|^2(2M_A^+ - M_A^{+2})}{|1-G|^2}\langle n_B, n_B\rangle
\end{aligned} \tag{8}
$$

If $h = M_A^+ = 1$, these expressions simplify to Equations 5.9-5.11 in [2].

3 try a different simplification

Instead of following the procedure from [3], we could set the matrix such that the contribution of the two PDs to the sensing of the correlated noise is equal, we would have $AM_A^+ = BM_B^+$. Allowing $A = hB$, and $M_B^+ = 2 - M_A^+$, this means:

$$
\begin{aligned}
M_B^+ &= \frac{2h}{h+1} \\
M_A^+ &= \frac{2}{h+1} \\
C &= ZB\frac{4h}{h+1} \\
G &= YZB\frac{4h}{h+1}
\end{aligned} \tag{9}
$$

The expressions for the power spectral densities 5 and cross power spectrum

6 become:

$$
\begin{aligned}
|1-G|^2\langle d_A, d_A\rangle &= h^2|ZB|^2\langle n_c, n_c\rangle + \left|1-\frac{YZB2h}{h+1}\right|^2\langle n_A, n_A\rangle + \left|\frac{YZB2h^2}{h+1}\right|^2\langle n_B, n_B\rangle \\
&= h^2|ZB|^2\langle n_c, n_c\rangle + \left|1-\frac{G}{2}\right|^2\langle n_A, n_A\rangle + \frac{h^2|G|^2}{4}\langle n_B, n_B\rangle \\
|1-G|^2\langle d_B, d_B\rangle &= |ZB|^2\langle n_c, n_c\rangle + \left|\frac{YZB2}{h+1}\right|^2\langle n_A, n_A\rangle + |1-YZhB2/(h+1)|^2\langle n_B, n_B\rangle \\
&= |ZB|^2\langle n_c, n_c\rangle + \frac{|G|^2}{4h^2}\langle n_A, n_A\rangle + \left|1-\frac{G}{2}\right|^2\langle n_B, n_B\rangle \\
|1-G|^2\langle d_A, d_B\rangle &= h|ZB|^2\langle n_c, n_c\rangle + \left[\frac{YZB2}{h+1} - \frac{|YZB|^2 4h}{(h+1)^2}\right]\langle n_A, n_A\rangle \\
&+ \left[Y^*Z^*B^*\frac{2h^2}{h+1} - |YZB|^2\frac{4h^3}{(h+1)^2}\right]\langle n_B, n_B\rangle \\
&= h|ZB|^2\langle n_C, n_C\rangle + \left[\frac{G}{2h} - \frac{|G|^2}{4h}\right]\langle n_A, n_A\rangle + \left[\frac{hG^*}{2} - \frac{h|G|^2}{4}\right]\langle n_B, n_B\rangle
\end{aligned} \tag{10}
$$

If we multiply d_B by h before taking spectra, using $\langle d_B, d_A\rangle = \langle d_A, b_B\rangle^*$, and substituting in the expression for the sensing function $4h^2|ZB|^2 = (h+1)^2|C|^2$, we can simplify to:

$$
\begin{aligned}
4|1-G|^2\langle d_A, d_A\rangle &= ((h+1)^2/4)|C|^2\langle n_c, n_c\rangle + |2-G|^2\langle n_A, n_A\rangle + h^2|G|^2\langle n_B, n_B\rangle \\
4|1-G|^2\langle hd_B, hd_B\rangle &= ((h+1)^2/4)|C|^2\langle n_c, n_c\rangle + |G|^2\langle n_A, n_A\rangle + h^2|2-G|^2\langle n_B, n_B\rangle \\
4|1-G|^2\langle d_A, hd_B\rangle &= ((h+1)^2/4)|C|^2\langle n_c, n_c\rangle + G(2-G^*)\langle n_A, n_A\rangle + h^2G^*(2-G)\langle n_B, n_B\rangle \\
4|1-G|^2\langle hd_B, d_A\rangle &= ((h+1)^2/4)|C|^2\langle n_c, n_c\rangle + G^*(2-G)\langle n_A, n_A\rangle + h^2G(2-G^*)\langle n_B, n_B\rangle
\end{aligned} \tag{11}
$$

These equations are very similar to Equations 5.9-5.11 in [2].

$$
\frac{(h+1)^2}{4}|C|^2\langle n_c, n_c\rangle = |2-G|^2\langle d_A, hd_B\rangle + |G|^2\langle hd_B, d_A\rangle - G(2-G^*)\langle d_A, d_A\rangle - G^*(2-G)\langle hd_B, hd_B\rangle \tag{12}
$$

We want to plot a comparison of this correlated noise with the DCPD SUM power spectral density:

$$
\begin{aligned}
\langle d_+, d_+\rangle &= \langle M_A^+ d_A + M_B^+ d_B, M_A^+ d_A + M_B^+ d_B, \rangle \\
&= \frac{1}{|1-G|^2}\left(|C|^2\langle n_c, n_c\rangle + \langle n_A, n_A\rangle + h^2\langle n_B, n_B\rangle\right)
\end{aligned} \tag{13}
$$

References

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[2] Craig Cahillane. *Controlling and Calibrating Interferometeric Gravitational Wave Detectors*. PhD thesis, California Institute of Technology (Pasadena, CA), 2021. https://dcc.ligo.org/DocDB/0149/P1800022/006/Cahillane_Craig_Thesis.pdf.

[3] Stefan Ballmer. Setting the dcpd matrix part ii lho alog 47217. https://alog.ligo-wa.caltech.edu/aLOG/index.php?callRep=47217, 2019.